Analytic Number Theory
Instructor: Ranjan Bera
M. Math Back-Paper Exam(2022).
Maximum marks: 80

Answer the following questions.

1) If $n \geq 1$ prove that

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

Define von Mangoldt function. Prove that for $n \geq 1$

$$\log n = \sum_{d|n} \Lambda(d).$$

4+1+3

2) Find the inverse of completely multiplicative function f. Is Liouville function completely multiplicative. Find the Bell series of Euler's totient function ϕ.

4+1+3

3) Define "Big oh" and "Small oh". State and prove weak and strong versions of Dirichlet asymptotic formulae for the partial sums of the divisor function $d(n)$.

2+6

4) Define the Chebyshev's functions ψ and θ. Let $a_1 < a_2 < \cdots < a_n < x$ be a set of positive integers such that no a_i divides the product of others. Prove that $n \leq \pi(x)$.

5) Define the Dirichlet character of a finite group. Prove that there is $\phi(k)$ distinct Dirichlet characters modulo k, each of which is completely multiplicative and periodic with period k. Find the Dirichlet characters for $k = 4$.

1+4+3

6) Define Ramanujan sum. Define induced modulus of Dirichlet character χ. Let χ be a Dirichlet character mod k and assume that $d|k$, $d > 0$. Then prove that d is an induced modulus for χ if, and only if,

$$\chi(a) = \chi(b)$$

whenever $(a, k) = (b, k) = 1$ and $a \equiv b \pmod 4$

1+1+6

7) Define Legendre's symbol. Prove that for all n

$$(n|p) \equiv n^{(p-1)/2} \pmod p$$

where p is odd prime. Determine the odd primes p for which 3 is a quadratic residue and those odd primes for which 3 is a quadratic nonresidue.

1+3+4

8) Using the index calculus solve $x^8 \equiv 8 \pmod{16}$. Assume that $\alpha \leq 3$, then prove that for every odd integer n there is a uniquely determine integer $b(n)$ such that

$$n \equiv (-1)^{(n-1/2)} 5^{b(n)} \pmod{2^\alpha}$$

with $1 \leq b(n) \leq \phi(2^\alpha)/2$

2+6

9) Define Hurwitz zeta function $\xi(s,a)$. Prove that Riemann Zeta function equal to 0 for every even negative number.

1+7

10) Assume that $\sum f(n)n^{-n}$ convergent absolutely for $\sigma > \sigma_a$. If f is multiplicative function prove that

$$\sum_{n=1}^{\infty} \prod_p \{1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^s} + \cdots\}$$

if $\sigma > \sigma_a$. Also prove that $\xi(s) = \prod_p \frac{1}{1-p^{-s}}$ if $\sigma > 1$

4+4